

Gravitational lensing in Tangherlini spacetime in the weak gravitational field and the strong gravitational field

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The gravitational lensing effects in the weak gravitational field by exotic lenses have been investigated intensively to find non-luminous exotic objects. Gravitational lenses in a strong gravitational field also are important since they are one of tests for general relativity, black holes and exotic objects. It is known that the light rays which pass just outside the photon sphere make faint images in the Schwarzschild spacetime and in wormhole spacetimes. This is the same in the Tangherlini spacetime. In this paper, we investigate the gravitational lensing effects in the Tangherlini spacetime in the weak gravitational field and the strong field limit. The gravitational lens model in the Tangherlini spacetime would work as a wide-range toy model for exotic lens models with the photon sphere since it is an exact solution of the Einstein equation in every dimension. We study the deflection angle of the light and the magnifications of images in the weak approximation and in the strong field limit. We derive the divergent part of the deflection angle in all dimensions and the regular part of the deflection angle in 4, 5 and 7 dimensions in the strong field limit, the deflection angle in all dimensions under the weak gravitational approximation and the relation between the size of the Einstein ring and the ones of the rings in the strong gravitational field. We also show that the images in the strong gravitational field are always fainter than the images in the weak gravitational field. We conclude that the images in the strong gravitational field have little effect on the total light curve and that the characteristic demagnification of the light curve will appear after considering the images in the strong gravitational field in higher dimensions. The gravitational lensing in the strong field limit in higher dimension would be related to the nature of the higher dimensional black hole.

I. INTRODUCTION

Gravitational lensing is useful to survey non-luminous objects for example extrasolar planets and dark matters (see [1–7] for the details of the gravitational lensing and references therein). The gravitational lensing effects of the Schwarzschild lens in the weak gravitational field have mainly investigated for a hundred years.

Exotic objects such as wormholes, cosmic strings also cause gravitational lensing effects. The gravitational lenses in wormhole spacetimes were pioneered by Kim and Cho [8] and Cramer *et al.* [9] and then the gravitational lensing effects in various wormhole spacetimes have been investigated [10–18] (See Visser [19] for the detail of wormholes). The Ellis wormhole [20, 21] is the simplest and earliest wormhole of the Morris-Thorne class [22, 23]. The gravitational lensing in the Ellis wormhole spacetime has been investigated intensively [10, 12, 17, 24–34] because of its simplicity and its features which are quite different from the ones of the mass lensing.

Recently, the gravitational lensing effects of exotic gravitational objects or a general spherical lens model including the singular isothermal sphere, the Schwarzschild lens and the Ellis wormhole in the weak gravitational field have been investigated. The demagnification of the light curves were studied by Kitamura *et al.* [35], the signed magnification sums were investigated by Tsukamoto and

Harada [27], the shear was investigated by Izumi *et al.* [36] and the microlensed image centroid motions were studied by Kitamura *et al.* [37]. These studies concentrate on the weak gravitational field and do not cover the gravitational lensing effects in the strong gravitational field.

About half a century ago, the images in the strong gravitational field were studied by Darwin [38, 39]. Darwin pointed out the existence of the relativistic images which are a series of faint images lying just outside the photon sphere [38] in the Schwarzschild spacetime. The countably infinite relativistic images are generally formed in spherically symmetric static spacetimes [34, 40, 41]. The gravitational lensing in the strong gravitational field by various black holes and wormholes has been investigated intensively in the recent decade (see [42–46] and references therein).

In this paper, we will investigate the gravitational lensing effects in the weak field approximation and in the strong field limit in the Tangherlini spacetime [47]. The Tangherlini lens model would work as a wide-range toy model for the exotic lens objects with strong gravitational field since the Tangherlini spacetime is a solution of the Einstein equations in all dimensions. The Tangherlini lens model is expected to show the general features of the gravitational lensing effects by exotic gravitational objects in both the weak and strong gravitational field.

The gravitational lens in the strong field limit is related to the other phenomena such as the quasi-normal modes of a black hole [48, 49] and the high-energy absorption cross section [50] which are caused by the na-

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ture of the null geodesic near the photon sphere. Thus, the investigation of gravitational lensing effects of the all-dimensional black hole in the strong field limit would give us a new perspective on the intrinsic property of the all-dimensional black hole.

This paper is organized as follows. In Sec. II, we review the null geodesic of the Tangherlini solution and investigate the deflection angle of light rays. In Secs. III and IV, we will investigate the deflection angle of the light in the weak field approximation and in the strong field limit, respectively. In Sec. V, we study the gravitational lens effects in the strong field limit in the Tangherlini spacetime. In Sec. VI, we conclude our results. In this paper we use the units in which the light speed $c = 1$ and Newton's constant $G = 1$.

II. DEFLECTION ANGLE OF LIGHT IN TANGHERLINI SPACETIME

In this section, we briefly review the null geodesic in the Tangherlini spacetime and investigate the deflection angle of light rays. The Tangherlini solution is given by [47]

$$ds^2 = - \left[1 - \left(\frac{r_g}{r} \right)^{d-3} \right] dt^2 + \frac{dr^2}{1 - \left(\frac{r_g}{r} \right)^{d-3}} + r^2 d\sigma_{d-2}^2, \quad (2.1)$$

where r_g is the event horizon radius and $d\sigma_{d-2}^2$ is

$$d\sigma_{d-2}^2 = d\theta_1^2 + \sum_{j=2}^{d-3} \prod_{i=1}^{j-1} \sin^2 \theta_i d\theta_j^2 + \prod_{i=1}^{d-3} \sin^2 \theta_i d\phi^2 \quad (2.2)$$

with the angular coordinates $\theta_i \in [0, \pi]$ and $\phi \in [0, 2\pi]$ and the integer i runs from 1 into $d-3$. The event horizon exists at $r = r_g$, where r_g is given by

$$r_g = \frac{16\pi M}{(d-2)A_{d-2}}, \quad (2.3)$$

where M is the black hole mass and A_{d-2} is the area of the unit sphere which is given by

$$A_{d-2} = \frac{2\pi^{\frac{d-1}{2}}}{\Gamma\left(\frac{d-1}{2}\right)}. \quad (2.4)$$

For stationarity and axial symmetry, there exist the Killing vectors $t^\mu \partial_\mu = \partial_t$ and $\phi^\mu \partial_\mu = \partial_\phi$, respectively.

Without loss of generality, we set $\sin \theta_i = 1$ and consider the induced line element

$$ds^2 = - \left[1 - \left(\frac{r_g}{r} \right)^n \right] dt^2 + \frac{dr^2}{1 - \left(\frac{r_g}{r} \right)^n} + r^2 d\phi^2, \quad (2.5)$$

where $n \equiv d-3$. From $k^\mu k_\mu = 0$, where k^μ is the photon wave number, the equation of the photon trajectory is obtained as

$$\left(\frac{dr}{d\phi} \right)^2 = r^4 G(r, b), \quad (2.6)$$

where

$$G(r, b) \equiv \frac{1}{b^2} - \frac{1}{r^2} + \frac{r_g^n}{r^{n+2}} \quad (2.7)$$

and $b \equiv L/E$ is the impact parameter of the photon and $E \equiv -g_{\mu\nu} t^\mu k^\nu$, and $L \equiv g_{\mu\nu} \phi^\mu k^\nu$ are the energy and the angular momentum of the photon, respectively. We assume that the conserved energy E is positive. We can assume $L > 0$ or $b > 0$ without loss of generality.

The equation $G(r, b) = 0$ has two positive solutions $r = r_-$ and r_0 for $b > b_c$, one positive solution $r = r_- = r_0$ for $b = b_c$ and no positive solution for $b < b_c$, where

$$b_c \equiv \left(\frac{n+2}{n} \right)^{\frac{1}{2}} \left(\frac{n+2}{2} \right)^{\frac{1}{n}} r_g \quad (2.8)$$

is the critical impact parameter. From Eqs. (2.6) and (2.7), we find that the photon is scattered if $b > b_c$ while it reaches the event horizon $r = r_g$ if $b < b_c$.

We will assume $b_c < b$ in what follows since we are interested in the scattering problem. Here we define r_0 as the larger solution of the equation $G(r, b) = 0$ i.e. $0 < r_- \leq r_0$. Thus, r_0 is the closest distance of a photon. From $G(r_0, b) = 0$, the relation between the impact parameter b and the closest distance r_0 is given by

$$\frac{1}{b^2} = \frac{1}{r_0^2} \left[1 - \left(\frac{r_g}{r_0} \right)^n \right]. \quad (2.9)$$

The derivative of $G(r, b)$ with respect to r is given by

$$\frac{\partial G(r, b)}{\partial r} = \frac{2}{r^3} - (n+2) \frac{r_g^n}{r^{n+3}}. \quad (2.10)$$

Thus, the radius of the photon sphere which satisfies $\partial G(r_m, b)/\partial r = 0$ is obtained as

$$r_m = \left(\frac{n+2}{2} \right)^{\frac{1}{n}} r_g. \quad (2.11)$$

The deflection angle α is given by

$$\alpha = I(b) - \pi, \quad (2.12)$$

where

$$I(b) \equiv 2 \int_{r_0}^{\infty} \frac{dr}{r^2 \sqrt{G(r, b)}}. \quad (2.13)$$

III. DEFLECTION ANGLE IN WEAK FIELD APPROXIMATION

In this section, we will calculate the deflection angle in the Tangherlini spacetime in weak field approximation by Keeton and Petters's method [51]. We define a small amount h by

$$h \equiv \left(\frac{r_g}{r_0} \right)^n \ll \left(\frac{r_g}{r_m} \right)^n = \frac{2}{n+2}, \quad (3.1)$$

because we assume $r_0 \gg r_m$ in this section. The relation between the impact parameter b and the closet distance r_0 (2.9) is expressed by

$$\left(\frac{r_0}{b}\right)^2 = 1 - h. \quad (3.2)$$

Thus,

$$h = \left(\frac{r_g}{b}\right)^n + O(h^2). \quad (3.3)$$

Using $x \equiv r_0/r$, the deflection angle α is given by

$$\alpha = 2 \int_0^1 \frac{dx}{\sqrt{1-x^2} \sqrt{1-hf(x)}} - \pi, \quad (3.4)$$

where

$$f(x) \equiv \frac{1-x^{n+2}}{1-x^2} = \frac{1+x+x^2+\dots+x^{n+1}}{1+x}. \quad (3.5)$$

$f(x)$ monotonically increases in the range of $0 \leq x \leq 1$ with the minimum value $f(0) = 1$ and the maximum value $f(1) = (n+2)/2$.

We will consider the Taylor series by the term of 1st degree

$$(1-hf(x))^{-\frac{1}{2}} = 1 + \frac{1}{2}hf(x) + O(h^2) \quad (3.6)$$

with respect to

$$hf(x) \ll \left(\frac{r_g}{r_m}\right)^n f(1) = 1. \quad (3.7)$$

Therefore, the deflection angle of the light is given by

$$\alpha = 2 \int_0^1 \frac{dx}{\sqrt{1-x^2}} + h \int_0^1 \frac{1-x^{n+2}}{(1-x^2)^{\frac{3}{2}}} dx - \pi + O(h^2). \quad (3.8)$$

We can easily integrate the first term,

$$\int_0^1 \frac{dx}{\sqrt{1-x^2}} = [\arcsin x]_0^1 = \frac{\pi}{2}. \quad (3.9)$$

Thus, the deflection angle is given by

$$\begin{aligned} \alpha &= H_{n+2}h + O(h^2) \\ &= H_{n+2} \left(\frac{r_g}{b}\right)^n + O\left(\left(\frac{r_g}{b}\right)^{2n}\right), \end{aligned} \quad (3.10)$$

where

$$H_m \equiv \int_0^{\frac{\pi}{2}} \frac{1 - \sin^m k}{\cos^2 k} dk, \quad (3.11)$$

where $k \equiv \arcsin x$ and m is a positive integer.

A recurrence formula is obtained as

$$H_{n+2} = H_n + B_n, \quad (3.12)$$

where B_n is

$$\begin{aligned} B_n &\equiv \int_0^{\frac{\pi}{2}} \sin^n k dk = \int_0^{\frac{\pi}{2}} \cos^n k dk \\ &= \frac{\sqrt{\pi}}{2} \frac{\Gamma\left(\frac{n+1}{2}\right)}{\Gamma\left(\frac{n+2}{2}\right)} \\ &= \begin{cases} \frac{(n-1)!!}{n!!} \frac{\pi}{2} & \text{for an even } n, \\ \frac{(n-1)!!}{n!!} & \text{for an odd } n. \end{cases} \end{aligned} \quad (3.13)$$

When n is even, we can put $n = 2L$ where L is a positive integer. From $H_2 = \pi/2$ and Eqs. (3.12) and (3.13), we obtain

$$H_{n+2} = H_2 + \sum_{m=1}^L B_{2m} = \frac{\pi}{2} \left[1 + \sum_{m=1}^L \frac{(2m-1)!!}{(2m)!!} \right] \quad (3.14)$$

Thus, the deflection angle is obtained as

$$\alpha = \frac{\pi}{2} \left[1 + \sum_{m=1}^L \frac{(2m-1)!!}{(2m)!!} \right] \left(\frac{r_g}{b}\right)^n + O\left(\left(\frac{r_g}{b}\right)^{2n}\right). \quad (3.15)$$

When n is odd, we can put $n = 2L - 1$. From $H_1 = 1$ and Eqs. (3.12) and (3.13), we get

$$H_{n+2} = H_1 + \sum_{m=1}^L B_{2m-1} = 1 + \sum_{m=1}^L \frac{(2m-2)!!}{(2m-1)!!}. \quad (3.16)$$

Here we have defined $0!! = 1$. Thus, the deflection angle is

$$\alpha = \left[1 + \sum_{m=1}^L \frac{(2m-2)!!}{(2m-1)!!} \right] \left(\frac{r_g}{b}\right)^n + O\left(\left(\frac{r_g}{b}\right)^{2n}\right). \quad (3.17)$$

We can also calculate the deflection angle by the non-linear terms with respect to h by Keeton and Petters's method [51].

IV. DEFLECTION ANGLE IN STRONG FIELD LIMIT

In this section, we will investigate the deflection angle in the Tangherlini spacetime in the strong field limit. We will express the deflection angle α of the light ray in the strong field limit by

$$\alpha(b) = -\bar{a} \log \left(\frac{b}{b_c} - 1 \right) + \bar{b} + O\left((b-b_c)^{\frac{1}{2}}\right), \quad (4.1)$$

or

$$\alpha(\theta) = -\bar{a} \log \left(\frac{\theta D_l}{b_c} - 1 \right) + \bar{b} + O\left((\theta D_l - b_c)^{\frac{1}{2}}\right) \quad (4.2)$$

where \bar{a} is a positive parameter, \bar{b} is a parameter, θ is the image angle and D_l is the distance between the observer and the lens object. For the small image angle $\theta \ll 1$, the impact parameter b can be described by

$$b = \theta D_l. \quad (4.3)$$

If we get the explicit expression for the deflection angle in the strong field limit, we can calculate a countably infinite number of relativistic images angle θ_N and the countably infinite magnifications μ_N individually [40].

We show the explicit expression for the divergent part of the deflection angle in the all-dimensional Tangherlini spacetime or the parameter \bar{a} and we integrate the regular part of the deflection angle in 4, 5 and 7 dimension.¹

Using by Eqs. (2.9) and

$$z \equiv 1 - \left(\frac{r_0}{r}\right)^n, \quad (4.4)$$

we rewrite $G(r, b)$ and $I(b)$ into $G(z, r_0)$ and $I(r_0)$, respectively, as follow:

$$G(z, r_0) = \frac{1}{r_0^2} \left\{ 1 - \left(\frac{r_g}{r_0}\right)^n + (1-z)^{\frac{2}{n}} \left[-1 + \left(\frac{r_g}{r_0}\right)^n (1-z) \right] \right\}. \quad (4.5)$$

$$I(r_0) = \int_0^1 R(z) f(z, r_0) dz, \quad (4.6)$$

where

$$R(z) \equiv \frac{2}{n} (1-z)^{\frac{1}{n}-1} \quad (4.7)$$

and

$$\begin{aligned} f(z, r_0) &\equiv \frac{1}{\sqrt{r_0^2 G(z, r_0)}} \\ &= \frac{1}{\sqrt{1 - \left(\frac{r_g}{r_0}\right)^n + (1-z)^{\frac{2}{n}} \left[-1 + \left(\frac{r_g}{r_0}\right)^n (1-z) \right]}}. \end{aligned} \quad (4.8)$$

We can expand $r_0^2 G(z, r_0)$ near the $z = 0$ and obtain

$$r_0^2 G(z, r_0) = \gamma(r_0)z + \beta(r_0)z^2 + \dots, \quad (4.9)$$

where

$$\gamma(r_0) \equiv \frac{1}{n} \left[2 - (n+2) \left(\frac{r_g}{r_0}\right)^n \right] \quad (4.10)$$

$$\beta(r_0) \equiv \frac{1}{n^2} \left[n - 2 + (n+2) \left(\frac{r_g}{r_0}\right)^n \right]. \quad (4.11)$$

Near the photon sphere $r_0 = r_m$, $\gamma(r_0)$ and $\beta(r_0)$ are expanded in a series,

$$\gamma(r_0) = \frac{2}{r_m} (r_0 - r_m) + O((r_0 - r_m)^2) \quad (4.12)$$

and

$$\beta(r_0) = \frac{1}{n} - \frac{2}{nr_m} (r_0 - r_m) + O((r_0 - r_m)^2) \quad (4.13)$$

respectively.

We will divide $I(r_0)$ into the divergent part $I_D(r_0)$ and the regular part $I_R(r_0)$ or

$$I(r_0) = I_D(r_0) + I_R(r_0). \quad (4.14)$$

The divergent part $I_D(r_0)$ is defined by

$$I_D(r_0) \equiv \int_0^1 R(0) f_0(z, r_0) dz, \quad (4.15)$$

where

$$f_0(z, r_0) \equiv \frac{1}{\sqrt{\gamma(r_0)z + \beta(r_0)z^2}}. \quad (4.16)$$

The divergent part $I_D(r_0)$ is calculated in a simple and straightforward way,

$$\begin{aligned} I_D(r_0) &= \frac{2}{n\sqrt{\beta(r_0)}} \log \left| \frac{\gamma(r_0) + 2\beta(r_0) + 2\sqrt{(\gamma(r_0) + \beta(r_0))\beta(r_0)}}{\gamma(r_0)} \right| \\ &= \frac{4}{n\sqrt{\beta(r_0)}} \log \left(\frac{\sqrt{\beta(r_0)} + \sqrt{\gamma(r_0) + \beta(r_0)}}{\sqrt{\gamma(r_0)}} \right). \end{aligned} \quad (4.17)$$

Therefore, the divergent part $I_D(r_0)$ in the strong field limit is obtained by

$$I_D(r_0) = -\frac{2}{\sqrt{n}} \log \left(\frac{r_0}{r_m} - 1 \right) + \frac{2}{\sqrt{n}} \log \frac{2}{n} + O(r_0 - r_m). \quad (4.18)$$

We will rewrite the divergent part $I_D(r_0)$ into a function $I_D(b)$ with respect to the impact parameter b since the lens equation is usually written as an equation in terms of the impact parameter b or the image angle θ . From the relation between the impact parameter b and the closet distance r_0 (2.9), we can regard the impact parameter $b(r_0)$ as a function with respect to the closet distance r_0 , we expand the impact parameter $b(r_0)$ in a series near $r_0 = r_m$ and we get

$$\begin{aligned} b(r_0) &= b_c + \frac{1}{2} \left(\frac{n+2}{n} \right)^{\frac{3}{2}} \frac{n}{r_m} (r_0 - r_m)^2 \\ &\quad + O((r_0 - r_m)^3). \end{aligned} \quad (4.19)$$

From Eqs. (2.8), (2.11) and (4.19) we obtain

$$\begin{aligned} \log \left(\frac{r_0}{r_m} - 1 \right) &= \frac{1}{2} \log \left(\frac{b}{b_c} - 1 \right) + \frac{1}{2} \log \left(\frac{2}{n+2} \right) \\ &\quad + O(r_0 - r_m). \end{aligned} \quad (4.20)$$

Hence, the divergent part is

$$\begin{aligned} I_D(b) &= -\frac{1}{\sqrt{n}} \log \left(\frac{b}{b_c} - 1 \right) + \frac{1}{\sqrt{n}} \log \frac{2(n+2)}{n^2} \\ &\quad + O\left((b - b_c)^{\frac{1}{2}}\right). \end{aligned} \quad (4.21)$$

¹ As below, we obey the convention of the analysis in the strong field limit but the definitions of some symbols such as z are different from the definitions by Bozza [40].

The regular part $I_R(r_0)$ is defined by

$$I_R(r_0) \equiv \int_0^1 g(z, r_0) dz, \quad (4.22)$$

where

$$g(z, r_0) \equiv R(z)f(z, r_0) - R(0)f_0(z, r_0). \quad (4.23)$$

We can expand $I_R(r_0)$ in powers of $(r_0 - r_m)$ and express it as a function $I_R(b)$ with respect to b as follow:

$$\begin{aligned} I_R(r_0) &= \sum_{l=0}^{\infty} \frac{1}{l!} (r_0 - r_m)^l \int_0^1 \left. \frac{\partial^l g}{\partial r_0^l} \right|_{r_0=r_m} dz \\ &= \frac{2}{n} \int_0^1 \left[\frac{\sqrt{n+2}(1-z)^{\frac{1}{n}-1}}{\sqrt{n-(1-z)^{\frac{2}{n}}(n+2z)}} - \frac{\sqrt{n}}{z} \right] dz \\ &\quad + O(r_0 - r_m) \\ &= 2\sqrt{n+2} \int_0^1 \frac{dy}{\sqrt{n-(n+2)y^2+2y^{n+2}}} \\ &\quad - \frac{2\sqrt{n}}{n} \int_0^1 \frac{dz}{z} + O\left((b-b_c)^{\frac{1}{2}}\right) \\ &= I_R(b), \end{aligned} \quad (4.24)$$

where we have used $y \equiv (1-z)^{\frac{1}{n}}$.

Thus, the deflection angle $\alpha(b)$ of the light on the Tangherlini spacetime in the strong field limit is obtained as

$$\begin{aligned} \alpha(b) &= I_D(b) + I_R(b) - \pi \\ &= -\frac{1}{\sqrt{n}} \log \left(\frac{b}{b_c} - 1 \right) + \frac{1}{\sqrt{n}} \log \frac{2(n+2)}{n^2} \\ &\quad + I_R(b) - \pi + O\left((b-b_c)^{\frac{1}{2}}\right). \end{aligned} \quad (4.25)$$

Hence, we get the parameters $\bar{a} = \frac{1}{\sqrt{n}}$ and $\bar{b} = \frac{1}{\sqrt{n}} \log \frac{2(n+2)}{n^2} + I_R(b) - \pi$.

We can analytically calculate the regular parts $I_R(b)$ for $n = 1, 2$ and 4 since the elliptic functions $I(b)$ for $n = 1, 2$ and 4 are integrable [33].

A. $n = 1$

We consider the case for $n = 1$. In this case, the critical impact parameter and the radius of the photon sphere are given by $b_c = \frac{3\sqrt{3}r_g}{2}$ and $r_m = \frac{3r_g}{2}$, respectively. The divergent part is obtained as

$$I_D(b) = -\log \left(\frac{b}{b_c} - 1 \right) + \log 6 + O\left((b-b_c)^{\frac{1}{2}}\right) \quad (4.26)$$

The regular part $I_R(b)$ is given by

$$\begin{aligned} I_R(b) &= 2 \int_0^1 \left(\frac{1}{z\sqrt{1-\frac{2}{3}z}} - \frac{1}{z} \right) dz + O\left((b-b_c)^{\frac{1}{2}}\right) \\ &= 2 \log \left[6 \left(2 - \sqrt{3} \right) \right] + O\left((b-b_c)^{\frac{1}{2}}\right). \end{aligned} \quad (4.27)$$

Thus, the deflection angle $\alpha(b)$ of the light is obtained as

$$\begin{aligned} \alpha(b) &= I_D(b) + I_R(b) - \pi \\ &= -\log \left(\frac{b}{b_c} - 1 \right) + \log \left[216 \left(7 - 4\sqrt{3} \right) \right] \\ &\quad - \pi + O\left((b-b_c)^{\frac{1}{2}}\right). \end{aligned} \quad (4.28)$$

Therefore, we get the parameters $\bar{a} = 1$ and $\bar{b} = \log \left[216 \left(7 - 4\sqrt{3} \right) \right] - \pi \simeq -0.40$. It recovers the deflection angle of the light in Schwarzschild spacetime in the strong field limit which was obtained by Bozza [40].

B. $n = 2$

For $n = 2$, the critical impact parameter and the radius of the photon sphere are $b_c = 2r_g$ and $r_m = \sqrt{2}r_g$, respectively. The divergent part is given by

$$\begin{aligned} I_D(b) &= -\frac{1}{\sqrt{2}} \log \left(\frac{b}{b_c} - 1 \right) + \frac{1}{\sqrt{2}} \log 2 \\ &\quad + O\left((b-b_c)^{\frac{1}{2}}\right). \end{aligned} \quad (4.29)$$

The regular part is obtained as

$$\begin{aligned} I_R(b) &= \int_0^1 \left(\frac{\sqrt{2}}{z\sqrt{1-z}} - \frac{\sqrt{2}}{z} \right) dz + O\left((b-b_c)^{\frac{1}{2}}\right) \\ &= 2\sqrt{2} \log 2 + O\left((b-b_c)^{\frac{1}{2}}\right). \end{aligned} \quad (4.30)$$

Thus, we obtain the deflection angle $\alpha(b)$ in the strong field limit for $n = 2$;

$$\begin{aligned} \alpha(b) &= -\frac{1}{\sqrt{2}} \log \left(\frac{b}{b_c} - 1 \right) + \frac{5\sqrt{2}}{2} \log 2 - \pi \\ &\quad + O\left((b-b_c)^{\frac{1}{2}}\right). \end{aligned} \quad (4.31)$$

In this case, the parameters are given by $\bar{a} = \frac{1}{\sqrt{2}}$ and $\bar{b} = \frac{5\sqrt{2}}{2} \log 2 - \pi \sim -0.69$.

C. $n = 4$

For $n = 4$, the critical impact parameter and the radius of the photon sphere are given by $b_c = \left(\frac{27}{4} \right)^{\frac{1}{4}} r_g$ and $r_m = 3^{\frac{1}{4}} r_g$, respectively. The divergent part is obtained by

$$\begin{aligned} I_D(b) &= -\frac{1}{2} \log \left(\frac{b}{b_c} - 1 \right) + \frac{1}{2} \log \frac{3}{4} \\ &\quad + O\left((b-b_c)^{\frac{1}{2}}\right). \end{aligned} \quad (4.32)$$

The regular part $I_R(b)$ is given by

$$\begin{aligned} I_R(b) &= 2\sqrt{3} \int_0^1 \frac{dy}{\sqrt{2-3y^2+y^6}} - \int_0^1 \frac{dz}{z} + O\left((b-b_c)^{\frac{1}{2}}\right) \\ &= \log 12 + O\left((b-b_c)^{\frac{1}{2}}\right) \end{aligned} \quad (4.33)$$

and the deflection angle in the strong field limit is obtained as

$$\alpha(b) = -\frac{1}{2} \log \left(\frac{b}{b_c} - 1 \right) + \log 6\sqrt{3} - \pi + O \left((b - b_c)^{\frac{1}{2}} \right) \quad (4.34)$$

and hence we get the parameter $\bar{a} = \frac{1}{2}$ and $\bar{b} = +\log 6\sqrt{3} - \pi \sim -0.80$.

V. GRAVITATIONAL LENSING

A. Lens equation

We consider the lens configuration which is given in Fig. 1. The light ray emitted by the source S bends near the lensing object L . The observer O does not see the source S with the source angle ϕ but the image I with the image angle θ . For simplicity, we assume that both the observer O and the source S are far from the lensing object L or $D_l \gg b$ and $D_{ls} \gg b$, where D_l and D_{ls} are the distance between the observer O and the lensing object L and between the lensing object L and the source object S . We also assume the thin lens approximation that the light ray bends on the lens plane. The impact parameter b of the light ray is described by $b = D_l \theta$. Under the assumptions, the effective deflection angle $\bar{\alpha}$, the source angle ϕ and the image angle θ are small or $|\bar{\alpha}| \ll 1$, $|\phi| \ll 1$ and $|\theta| \ll 1$. The effective deflection angle $\bar{\alpha}$ is defined by

$$\bar{\alpha} \equiv (\alpha \bmod 2\pi). \quad (5.1)$$

The deflection angle α is expressed by

$$\alpha = \bar{\alpha} + 2\pi N, \quad (5.2)$$

where N is a non-negative integer which denotes the winding number of the light ray.

Then, the lens equation is given by

$$D_{ls}\bar{\alpha} = D_s(\theta - \phi), \quad (5.3)$$

where D_s is the separation between the observer O and the source S and satisfies the relation $D_s = D_l + D_{ls}$. If the source angle $\phi = 0$, ring-shaped images which are called the Einstein ring with the angle θ_0 for $N = 0$ and the relativistic Einstein ring with the angle $\theta_{N \geq 1}$ for $N \geq 1$ appear from the symmetry. From $N = 0$, $\phi = 0$ and Eqs. (3.10), (4.3), (5.2) and (5.3), the Einstein ring angle is given by

$$\theta_0 \sim \left(H_{n+2} \frac{D_{ls}}{D_s} \right)^{\frac{1}{n+1}} \left(\frac{r_g}{D_l} \right)^{\frac{n}{n+1}}. \quad (5.4)$$

The behaviors of the Tangherlini lens model in the weak field approximation have been known already because the lens model is included in the exotic lens model or the general spherical lens model [27, 35–37]. Here, we refer only to the image angles and the magnification

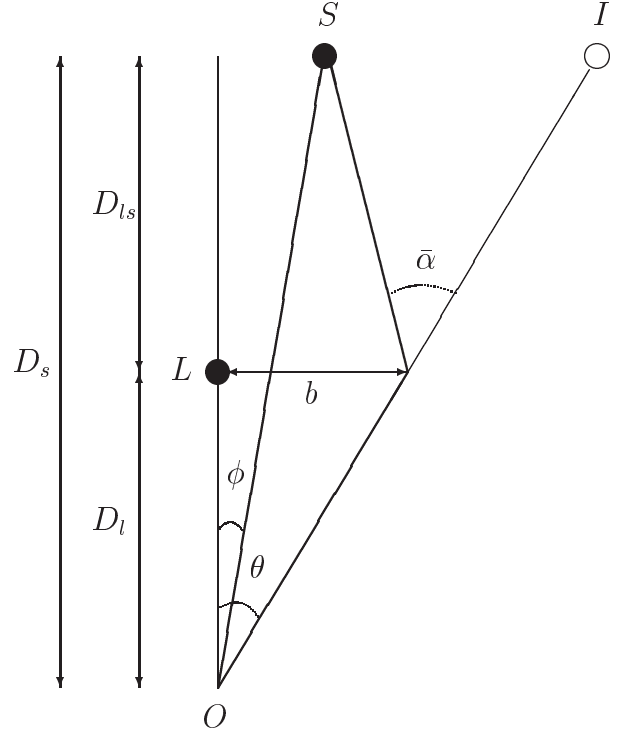


FIG. 1. The configuration of the gravitational lensing. The light ray emitted by the source object S bends on the lens plane and the observer O does not observe the source S with the source angle ϕ but the image I with the image angle θ . $\bar{\alpha}$ is the effective deflection angle and b is the impact parameter of the light. D_l and D_{ls} are the distances between the observer O and the lens object L and between the lens object L and the source object S , respectively. The distance between the observer O and the source S is given by $D_s = D_l + D_{ls}$.

in the directly aligned limit ($|\phi| \ll \theta_0 \ll 1$). Under the weak approximation, the lens equation has two solutions θ_{0+} and θ_{0-} regardless of the source angle ϕ for $n \geq 1$. For $|\phi| \ll \theta_0 \ll 1$, the image angles $\theta_{0\pm}$ and the magnification $\mu_{0\pm}$ are given by [27, 35]

$$\theta_{0\pm} = \pm\theta_0 + \frac{\phi}{1+n} \pm \frac{n\phi^2}{2(1+n)^2\theta_0} + O \left(\frac{\phi^3}{\theta_0^2} \right) \quad (5.5)$$

and

$$\mu_{0\pm} \sim \frac{1}{1+n} \frac{\phi \pm \theta_0}{\phi}, \quad (5.6)$$

respectively. The total magnification μ_0 in the directly aligned limit is given by

$$\mu_0 \equiv |\mu_{0+}| + |\mu_{0-}| = \frac{2}{1+n} \frac{\theta_0}{\phi}. \quad (5.7)$$

The relativistic Einstein rings or the relativistic images always appear on the region just outside the photon sphere. The angle of the innermost relativistic Einstein

ring is obtained as

$$\theta_\infty = \frac{b_c}{D_l} = \left(1 + \frac{2}{n}\right)^{\frac{1}{2}} \left(1 + \frac{n}{2}\right)^{\frac{1}{n}} \frac{r_g}{D_l}. \quad (5.8)$$

The relation between the Einstein ring angle θ_0 , the relativistic Einstein ring angle θ_∞ and the relativistic image angle $\theta_{N \geq 1}(\phi)$ is obtained by

$$\begin{aligned} \theta_{N \geq 1}(\phi) &\sim \theta_\infty \\ &\sim \sqrt{\frac{n+2}{n}} \left(\frac{n+2}{2H_{n+2}} \frac{D_s}{D_{ls}} \right)^{\frac{1}{n}} \theta_0^{\frac{n+1}{n}}. \end{aligned} \quad (5.9)$$

B. Magnifications and Images of the Relativistic Images

We will briefly summarize the magnifications $\mu_{N \geq 1}$ and the angles $\theta_{N \geq 1}$ of the relativistic images [40, 52] and investigate them in the Tangherlini spacetime. We use the deflection angle $\alpha(\theta)$ (4.2) in this subsection.

When the winding number $N \geq 1$, we define an angle $\theta_{N \geq 1}^0$ by

$$\alpha(\theta_{N \geq 1}^0) = 2\pi N. \quad (5.10)$$

From Eqs. (4.2) and (5.10), we obtain

$$\theta_{N \geq 1}^0 = \frac{b_c}{D_l} \left[1 + e^{(\bar{b}-2\pi N)\sqrt{n}} \right]. \quad (5.11)$$

We expand the deflection angle $\alpha(\theta)$ around $\theta = \theta_{N \geq 1}^0$ to obtain the effective deflection angle $\bar{\alpha}$. We define a small angle

$$\Delta\theta_{N \geq 1} \equiv \theta_{N \geq 1}(\phi) - \theta_{N \geq 1}^0, \quad (5.12)$$

where $\theta_{N \geq 1}(\phi)$ is the solution of the lens equation (5.3) or the relativistic image angle with the winding number $N \geq 1$. From Eqs. (4.2) and (5.11), the effective deflection angle of the light in the strong field limit is given by

$$\bar{\alpha} = -\frac{D_l}{b_c} \frac{e^{\sqrt{n}(-\bar{b}+2\pi N)}}{\sqrt{n}} \Delta\theta_{N \geq 1}. \quad (5.13)$$

We substitute the effective deflection angle (5.13) into the lens equation (5.3) to obtain

$$\phi = \theta_{N \geq 1}^0 + \left[1 + \frac{D_l}{b_c} \frac{D_{ls}}{D_s} \frac{e^{\sqrt{n}(-\bar{b}+2\pi N)}}{\sqrt{n}} \right] \Delta\theta_{N \geq 1}. \quad (5.14)$$

From Eqs. (5.12) and (5.14), we get the the relativistic image angle $\theta_{N \geq 1}(\phi)$

$$\theta_{N \geq 1}(\phi) \simeq \theta_{N \geq 1}^0 + \frac{b_c}{D_l} \frac{D_s}{D_{ls}} \sqrt{n} e^{\sqrt{n}(\bar{b}-2\pi N)} (\phi - \theta_{N \geq 1}^0), \quad (5.15)$$

where we have used $b_c/D_l \ll 1$. From Eqs. (5.11) and (5.15), the innermost relativistic image angle is obtained as

$$\theta_\infty = \theta_\infty^0 = \frac{b_c}{D_l}. \quad (5.16)$$

From Eqs. (5.11), (5.15) and (5.16), the difference of the angles between the outermost relativistic image and innermost one is given by

$$\theta_1 - \theta_\infty \simeq \theta_1^0 - \theta_\infty = \theta_\infty e^{\sqrt{n}(\bar{b}-2\pi)}. \quad (5.17)$$

The magnification $\mu_{N \geq 1}$ of the relativistic image is obtained as

$$\begin{aligned} \mu_{N \geq 1} &\simeq \frac{\theta_{N \geq 1}}{\phi} \frac{d\theta_{N \geq 1}}{d\phi} \bigg|_{\theta_{N \geq 1} = \theta_{N \geq 1}^0} \\ &\simeq \frac{1}{\phi} \frac{b_c^2}{D_l^2} \frac{D_s}{D_{ls}} \sqrt{n} e^{\sqrt{n}(\bar{b}-2\pi N)}. \end{aligned} \quad (5.18)$$

The sum of the magnifications of all the relativistic images is given by

$$\begin{aligned} \sum_{N=1}^{\infty} \mu_N &\simeq \mu_1 \simeq \frac{1}{\phi} \frac{b_c^2}{D_l^2} \frac{D_s}{D_{ls}} \sqrt{n} e^{\sqrt{n}(\bar{b}-2\pi)} \\ &\simeq \frac{2}{H_{n+2}^{\frac{2}{n}} \sqrt{n} \phi} \left[\frac{(n+2)D_s}{2D_{ls}} \right]^{\frac{2}{n}+1} \theta_0^{\frac{2n+2}{n}} e^{\sqrt{n}(\bar{b}-2\pi)}. \end{aligned} \quad (5.19)$$

The sum of the magnification of all the relativistic images become unity when the source angle is

$$\phi \simeq \frac{2}{H_{n+2}^{\frac{2}{n}} \sqrt{n}} \left[\frac{(n+2)D_s}{2D_{ls}} \right]^{\frac{2}{n}+1} \theta_0^{\frac{2n+2}{n}} e^{\sqrt{n}(\bar{b}-2\pi)}. \quad (5.20)$$

In the directly aligned limit, the ratio of the magnification of the weak field image divided by the sum of the magnification of all the relativistic images is given by

$$\frac{\mu_0}{\sum_{N=1}^{\infty} \mu_N} \simeq \frac{\mu_0}{\mu_1} \simeq \frac{H_{n+2}^{\frac{2}{n}} \sqrt{n}}{n+1} \left[\frac{2D_{ls}}{(n+2)D_s \theta_0} \right]^{\frac{2}{n}+1} e^{\sqrt{n}(2\pi-\bar{b})}. \quad (5.21)$$

The ratio shows that the relativistic images are always fainter than images in the weak field. Thus, we can ignore the effect of the relativistic images on the light curve. However, this does not mean that we cannot observe the relativistic images separately since they can get bright when the source angle is small.

The sum of the magnifications of the relativistic images excluding the outermost relativistic image is given by

$$\sum_{N=2}^{\infty} \mu_N \simeq \mu_2 \simeq \frac{1}{\phi} \frac{b_c^2}{D_l^2} \frac{D_s}{D_{ls}} \sqrt{n} e^{\sqrt{n}(\bar{b}-4\pi)}. \quad (5.22)$$

From Eqs. (5.22) and (5.18), the ratio of the magnification of the outermost relativistic image divided by the sum of the magnification of the other relativistic images is given by

$$\frac{\mu_1}{\sum_{N=2}^{\infty} \mu_N} \simeq \frac{\mu_1}{\mu_2} \simeq e^{\sqrt{n}2\pi}. \quad (5.23)$$

VI. SUMMARY

We investigated the gravitational lensing effects in the Tangherlini spacetime in the weak gravitational field and in the strong field limit. The Tangherlini lens would work as a wide-range toy model for an exotic lens model or a general spherical lens model [27, 35–37] with a photon sphere. The gravitational lensing in the strong field limit in higher dimension would be related to the nature of the higher dimensional black hole such as quasi-normal modes of black hole [48, 49] and high-energy absorption cross section [50].

We derived the divergent part of the deflection angle in all dimensions and the regular part in 4, 5 and 7 dimensions in the strong field limit, the deflection angle in all dimensions under the weak gravitational approxima-

tion and the relation between the size of the Einstein ring and the ones of the relativistic Einstein rings in all dimensions. We also shown that the relativistic images are always fainter than the images in the weak gravitational field.

Kitamura *et al.* [35] studied the demagnification of the light curves of the exotic lens object in the weak gravitational field. We conclude that the images in the strong gravitational field have little effect on the total light curve and that the characteristic demagnification of the light curve will appear after considering the images in the strong gravitational field for $n > 1$.

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